

FIXED POINT THEOREMS FOR A CLASS OF MAPPINGS DEPENDING OF ANOTHER FUNCTION AND DEFINED ON CONE METRIC SPACES

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ABSTRACT. In this paper we study the existence and uniqueness of fixed points of a class of mappings defined on complete, (sequentially compact) cone metric spaces, without continuity conditions and depending on another function.

1. INTRODUCTION

Since the Banach's contraction principle come out in 1922 (see, e.g., [13], [17]), several type of contraction mappings, acting on metric spaces, have been appearing, at point that in 1977 B.E. Rhoades [35] made a comparison of 250 different type (or classes) of contraction mappings. Chatterjea, Eldestein, Hardy & Rogers, Kannan and Shimi -contraction type of maps, are some of the most extensively studied [7, 14, 15, 19, 20, 31, 32]. Generalizations of such classes are already known, we quote here only [4], [8], [10], [29] and [30] for some of such generalizations. Also, there exist a lot of works devoted to study the relations among these (and others) generalized contraction-type maps, see for instance [9, 36, 37]. Recently, a new generalization of contraction mappings acting on complete metric spaces was introduced by A. Beiranvand, S. Moradi, M. Omid and H. Pazandeh [3] called: T -contraction and T -contrative which are contraction mappings depending on another function T , extending in this fashion the Banach's contraction principle and the Edelstein's fixed point Theorem. Furthermore, in [26] S. Moradi define the corresponding T -Kannan contraction.

The class of contraction mappings that will be play a principal role in this paper is the introduced in 1986 by L. Nova [23], (or see [11], [12] for instance) which is defined as follows:

Let (M, d) be a complete metric space, $K \subset M$ closed and $T : K \longrightarrow K$ an arbitrary mapping. We shall say T belong or is of class $D(a, b)$ when T satisfies the following condition, for $a, b \geq 0$ and any $x, y \in K$.

$$(1.1) \quad d(Tx, Ty) \leq ad(x, y) + b[d(x, Tx) + d(y, Ty)].$$

Fixed point's existence and uniqueness theorems, as well examples of mappings on $D(a, b)$, can be found in [11, 12, 23, 27]. Notice that a mapping T satisfying (1.1) is

a Banach contraction map when $b = 0$ and $a < 1$. In addition, for $a = 0$ we obtain that T is a Kannan contraction and for $a \neq 0$, a Reich contraction for the special case of $a_2 = a_3$, while any mapping, continuous or not, is in class $D(1, 1)$.

On the other hand, generalized Banach spaces, specially ordered Banach spaces, bring a large variety of spaces where contraction type of mappings can be defined, providing conditions for the existence and uniqueness of fixed points (see, [5, 6, 26, 38]). Using this approach, Huan Long - Guang and Zhang Xian in [22], introduced the concept of cone metric space, where the set of real numbers is replaced by an ordered Banach space. They introduced the basic definitions and discuss some properties of convergence of sequences in cone metric spaces. Ordered Banach spaces and normal cones are especially useful in applications on optimization theory coming from non-convex analysis, see for instance [24, 25].

In the same paper [22], several fixed point theorems for contractive single-valued maps (Banach, Eldestein and Kannan type-mappings) in such spaces was proved. Subsequently, some other mathematicians, (see, e.g., [1, 2, 16, 18, 29, 34]), have generalized these results of Guang and Zhang. The corresponding version of $D(a, b)$ defined on cone metric spaces was given in 2008 by M.S. Khan and M. Samanipour [21] as a particular case of the class $E(a, b, c)$ introduced there (Definition 2.4).

The main goal of this paper is, using the approach of A. Beiranvand *et al* [3, 26] and M.S. Khan and M. Samanipour [21], to introduce a generalization of the class $D(a, b)$ and proof the corresponding theorems of existence and uniqueness of fixed points.

2. CONE METRIC SPACES

Consistent with Guang and Zhang [22], we recall the definitions of cone metric space, the notion of convergence and other results that will be needed in the sequel.

Let E be a real Banach space and P a subset of E . P is called a *cone* if and only if:

- (1) P is non-empty, closed and $P \neq \{0\}$;
- (2) $a, b \in \mathbb{R}$, $a, b \geq 0$ and $x, y \in P \Rightarrow ax + by \in P$;
- (3) $x \in P$ and $-x \in P \Rightarrow x = 0$.

For a given cone $P \subseteq E$, we can define a partial ordering \leq on E with respect to P by

$$x \leq y, \quad \text{if and only if} \quad y - x \in P.$$

We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stands for $y - x \in \text{Int } P$, where $\text{Int } P$ denotes the interior of P . The cone $P \subset E$ is called *normal* if there is a number $K > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y, \quad \text{implies} \quad \|x\| \leq K\|y\|.$$

The least positive number satisfying inequality above is called the *normal constant* of P .

The cone P is called *regular* if every increasing sequence which is bounded from above is convergent. That is, if (x_n) is a sequence such that

$$x_1 \leq x_2 \leq \dots \leq x_n \leq \dots \leq y$$

for some $y \in E$, then there is $x \in E$ such that $\|x_n - x\| \rightarrow 0$, $(n \rightarrow \infty)$.

In [34] was proved that every regular cone is normal, also other properties of the cones can be found there.

In the following we always suppose E is a real Banach space, P is a cone with $\text{Int } P \neq \emptyset$ and \leq is a partial ordering with respect to P .

Definition 2.1 ([22]). Let M be a nonempty set. Suppose the mapping $d : M \times M \longrightarrow E$ satisfies:

- (1) $0 < d(x, y)$ for all $x, y \in M$ and $d(x, y) = 0$ if and only if $x = y$;
- (2) $d(x, y) = d(y, x)$ for all $x, y \in M$;
- (3) $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in M$.

Then d is called a *cone metric* on M and (M, d) is called a *cone metric space*.

It is obvious that cone metric spaces generalize metric spaces.

Examples 2.2. (1) ([22, Example 1]) Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\} \subset \mathbb{R}^2$, $M = \mathbb{R}$ and $d : M \times M \longrightarrow E$ such that

$$d(x, y) = \begin{pmatrix} |x - y|, & \alpha|x - y| \end{pmatrix}$$

where $\alpha \geq 0$ is a constant. Then (M, d) is a cone metric space.

- (2) ([34, Example 2.3]) Let $E = (C^2[0, 1], \mathbb{R})$, with the norm

$$\|f\| := \|f\|_\infty + \|f'\|_\infty,$$

and the cone $P = \{\varphi \in E : \varphi \geq 0\}$. This is an example of a non-normal cone.

Definition 2.3 ([22]). Let (M, d) be a cone metric space and (x_n) a sequence in M . Then:

- (1) (x_n) converges to $x \in M$ if for every $c \in E$, with $0 \ll c$, there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$d(x_n, x) \ll c.$$

We denote this by $\lim_{n \rightarrow \infty} x_n = x$ or $x_n \longrightarrow x$, $(n \rightarrow \infty)$.

- (2) If for any $c \in E$, there is a number $n_0 \in \mathbb{N}$ such that for all $m, n \geq n_0$

$$d(x_n, x_m) \ll c,$$

then (x_n) is called a *Cauchy sequence* in M ;

- (3) (M, d) is a *complete cone metric space* if every Cauchy sequence is convergent in M .

Definition 2.4. Let (M, d) be a cone metric space. If for any sequence (x_n) in M , there is a subsequence (x_{n_i}) of (x_n) such that (x_{n_i}) is convergent in M , then (M, d) is called a *sequentially compact cone metric space*.

Next Definition and subsequent Lemma are given in [3] in the scope of metric spaces, here we will rewrite it in terms of cone metric spaces.

Definition 2.5. Let (M, d) be a cone metric space. Then

- (1) T is said to be *continuous* if $\lim_{n \rightarrow \infty} x_n = x$, implies that $\lim_{n \rightarrow \infty} Tx_n = Tx$ for every (x_n) in M ;
- (2) T is said to be *sequentially convergent* if we have, for every sequence (y_n) , if $T(y_n)$ is convergent, then (y_n) is also convergent;

- (3) T is said to be *subsequentially convergent* if we have, for every sequence (y_n) , if $T(y_n)$ is convergent, then (y_n) has a convergent subsequence.

Lemma 2.1. *Let (M, d) be a sequentially compact cone metric space. Then every function $T : M \longrightarrow M$ is subsequentially convergent and every continuous function $T : M \longrightarrow M$ is sequentially convergent.*

3. THE CLASS $D_T(a, b)$

In this section, we will introduce a class of mappings that generalize the class $D(a, b)$ defined in [23] and [21], for Banach spaces and complete metric cone spaces respectively. As in [34] our fixed point theorems for elements in this class will be proved omitting the assumption of normal cone.

Definition 3.1. Let (M, d) be a complete cone metric space and $T, S : M \longrightarrow M$ two functions such that T is continuous and injective. A mapping S is of class $D_T(a, b)$, for $a, b \geq 0$ constants, if satisfies the following inequality

$$(3.2) \quad d(TSx, TSy) \leq ad(Tx, Ty) + b[d(Tx, TSx) + d(Ty, TSy)]$$

for all $x, y \in M$.

Example 3.2. We are going to consider a function $S \notin D(a, b)$ and a function T , injective and continuous, such that $S \notin D_T(a, b)$.

Let $E = (C[0, 1], \mathbb{R})$, the cone $P = \{\varphi \in E : \varphi \geq 0\}$, $M := [0, 1]$ and $d(x, y) = |x - y|e^t$ where $e^t \in E$. Let consider $Sx = \sqrt{x}$, the continuous and injective function $Tx = \alpha x$, $\alpha \in \mathbb{R} \setminus \{0\}$. First we will prove that for $0 \leq a, b \leq 1$, $S \notin D(a, b)$. In fact, if we suppose the opposite, then

$$d(Sx, Sy) \leq ad(x, y) + b[d(x, Tx) + d(y, Ty)].$$

Rewriting this inequality we obtain

$$|Sx - Sy|e^t \leq a|x - y|e^t + b[|x - Sx| + |y - Sy|]e^t,$$

from last inequality we conclude that $S \in D(a, b)$ in the sense of Banach spaces, which means that $S \in D(a, b)$ for the Banach space $([0, 1], |\cdot|)$. But Example 1 on [12], proves that $S \notin D(a, b)$ for $0 \leq a, b \leq 1$, proving in this way that in fact $S \notin D(a, b)$ when the mapping is defined in the cone metric space E . On the other hand, notice that if $S \in D_T(a, b)$ then next inequality follows

$$d(\alpha\sqrt{x}, \alpha\sqrt{y}) \leq ad(\alpha x, \alpha y) + b[d(\alpha x, \alpha\sqrt{x}) + d(\alpha y, \alpha\sqrt{y})]$$

or, that $S \in D(a, b)$ which is false.

Example 3.3. Now, we will consider a function $S \notin D(a, b)$ and an injective, continuous function T such that $S \in D_T(a, b)$.

In this case $E = (C[0, 1], \mathbb{R})$, the cone $P = \{\varphi \in E : \varphi \geq 0\}$, $M := [\frac{1}{2}, 1]$ and $d(x, y) = |x - y|e^t$ where $e^t \in E$, consider the function S define as $Sx = \sqrt{x}$ which does not belong to $D(a, b)$ for $0 \leq a, b \leq 1$. Let the continuous and injective function $Tx = \ln x$. We will prove that $S \in D_T(a, b)$. For attain a such goal we need to proof that the following inequality holds.

$$|\ln \sqrt{x} - \ln \sqrt{y}| \leq a|\ln x - \ln y| + b[|\ln x - \ln \sqrt{x}| + |\ln y - \ln \sqrt{y}|].$$

Or equivalently that

$$(3.3) \quad \left| \ln \sqrt{\frac{x}{y}} \right| \leq a \left| \ln \frac{x}{y} \right| + b \left[\left| \ln \frac{\sqrt{x}}{x} \right| + \left| \ln \frac{y}{\sqrt{y}} \right| \right].$$

Notice that inequality above holds for $x = y$. So, we can assume without loss of generality that $x > y$. Then (3.3) can be rewrite as

$$\ln \sqrt{\frac{x}{y}} \leq a \ln \frac{x}{y} + b \ln \frac{\sqrt{x}}{x} - b \ln \frac{y}{\sqrt{y}}.$$

Since $0 \leq a, b < 1$ and $x, y \in M$, then we have that last inequality is satisfies if and only if

$$\sqrt{\frac{x}{y}} \leq e^{a+b} \frac{x}{y} \frac{\sqrt{x}}{x} \frac{y}{\sqrt{y}}$$

which is holds if and only if $a + b \geq 0$. Therefore, $S \in D_T(a, b)$.

Example 3.4. In this example we will show a function $S \in D(a, b)$ such that for a continuous and injective function T , we get $S \notin D_T(a, b)$.

Let $E = (C[0, 1], \mathbb{R})$, the cone $P = \{\varphi \in E : \varphi \geq 0\}$, $M := [0, 1]$ and $d(x, y) = |x - y|e^t$ where $e^t \in E$. In this case let the function $Sx = x$. We have that $S \in D(a, b)$ for $a < 1$. Take $Tx = \sqrt{x}$. Then if $S \in D_T(a, b)$, the following inequality is satisfied

$$d(\sqrt{x}, \sqrt{y}) \leq ad(\sqrt{x}, \sqrt{y}),$$

rewriting inequality above we get

$$|\sqrt{x} - \sqrt{y}| \leq a|\sqrt{x} - \sqrt{y}|,$$

which is false because $a < 1$.

The following result describe the asymptotic behavior of the elements on $D_T(a, b)$, also give conditions for existence and uniqueness of fixed points for applications in that class.

Theorem 3.1. *Let (M, d) be a complete cone metric space, P be a cone, and let $T, S : M \longrightarrow M$ be mappings such that T is continuous and injective. If $S \in D_T(a, b)$, $a, b \geq 0$ and moreover:*

(i) *If $a + 2b < 1$, then*

$$\lim_{m, n \rightarrow \infty} d(TS^m x_0, TS^n x_0) = 0, \text{ for every } x_0 \in X.$$

(ii) *If $a + 2b < 1$, and T is subsequentially convergent. Then S has a fixed point.*

(iii) *If $a < 1$, then the fixed point of S is unique.*

(iv) *If $a + 2b < 1$ and furthermore T is sequentially convergent, then the sequence of iterates $(S^n x_0)$, for every $x_0 \in X$, converges to the unique fixed point of S .*

(v) *If $b < 1$ and $(x_n)_n$ is a sequence of points on X converging to a fixed point of S , then*

$$(3.4) \quad d(Tx_n, TSx_n) \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

(vi) *If $b < 1$, $(x_n)_n$ converges to a point p and (3.4) holds, then p is a fixed point of S .*

(vii) *If $b < 1$ and S has a fixed point p , then S is continuous at p .*

Proof. (i) Let x_0 be an arbitrary point, we define the iterative sequence $(x_n)_n$ by setting $x_{n+1} := Sx_n$ (equivalently, $x_n := S^n x_0$), $n = 1, \dots$. Using the fact that $S \in D_T(a, b)$, we have

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &= d(TSx_{n-1}, TSx_n) \\ &\leq ad(Tx_{n-1}, Tx_n) + b[d(Tx_{n-1}, TSx_{n-1}) + d(Tx_n, TSx_n)] \end{aligned}$$

so that,

$$d(Tx_n, Tx_{n+1}) \leq \frac{a+b}{1-b} d(Tx_{n-1}, Tx_n).$$

Repeating the argument we obtain

$$d(Tx_n, Tx_{n+1}) \leq \lambda^n d(Tx_0, Tx_1),$$

where $\lambda := \frac{a+b}{1-b} < 1$. From inequality above, we have for every $m, n \in \mathbb{N}$ such that $m > n$

$$\begin{aligned} d(Tx_m, Tx_n) &\leq d(Tx_m, Tx_{m-1}) + d(Tx_{m-1}, Tx_{m-2}) \\ &\quad + \cdots + d(Tx_{n+1}, Tx_n) \\ &\leq [\lambda^{m-1} + \lambda^{m-2} + \cdots + \lambda^n] d(Tx_0, Tx_1) \\ &\leq [\lambda^n + \lambda^{n-1} + \cdots + 1] d(Tx_0, Tx_1) = \frac{\lambda^n}{1-\lambda} d(Tx_0, Tx_1). \end{aligned}$$

Let $0 \ll c$ be given. Choose a natural number N_1 such that $\frac{\lambda^n}{1-\lambda} d(Tx_0, Tx_1) \ll c$, for all $n \geq N_1$. Thus

$$d(Tx_m, Tx_n) \ll c, \quad m > n.$$

So, we have that $d(Tx_m, Tx_n) \ll \frac{c}{n}$ for all $n \geq 1$. Therefore, $\frac{c}{n} - d(Tx_m, Tx_n) \in P$ for all $n \geq 1$. Since $\frac{c}{n} \rightarrow 0$ ($n \rightarrow \infty$) and P is closed, then $-d(Tx_m, Tx_n) \in P$. Thus

$$\lim_{n, m \rightarrow \infty} d(TS^m x_0, TS^n x_0) = 0.$$

- (ii) From part (i) we know that $(Tx_n)_n$ is a Cauchy sequence in (X, d) . Since (X, d) is a complete cone metric space, there exists $x \in X$ such that

$$(3.5) \quad \lim_{n \rightarrow \infty} Tx_n = x.$$

On the other hand, the hypothesis that T is a subsequentially convergent mapping, imply that $(x_n)_n$ has a convergent subsequence. So, there exists $y \in X$ and $(x_{n(k)})_k$ such that $\lim_{k \rightarrow \infty} x_{n(k)} = y$. From the continuity of T we have that $\lim_{k \rightarrow \infty} Tx_{n(k)} = Ty$ and by equality (3.5) we get $Ty = x$. Now, choose a natural number N_2 such that $d(Ty, Tx_{n(k)-1}) \ll \frac{1-b}{4a}c$, $\lambda^{n(k)-1} d(TSx_0, Tx_0) \ll \frac{1-b}{4b}c$, $\lambda^{n(k)} d(TSx_0, Tx_0) \ll \frac{1-b}{4}c$ and $d(Tx_{n(k)+1}, Ty) \ll \frac{1-b}{4}c$, for all $n(k) > N_2$.

$$\begin{aligned} d(TSy, Ty) &\leq d(TSy, TS^{n(k)}x_0) + d(TS^{n(k)}x_0, TS^{n(k)+1}x_0) \\ &\quad + d(TS^{n(k)+1}x_0, Ty) \\ &\leq ad(Ty, TS^{n(k)-1}x_0) + b[d(Ty, TSy) \\ &\quad + d(TS^{n(k)-1}x_0, TS^{n(k)}x_0)] \\ &\quad + d(TS^{n(k)}x_0, TS^{n(k)+1}x_0) + d(TS^{n(k)+1}x_0, Ty) \\ &\leq ad(Ty, TS^{n(k)-1}x_0) + b[d(Ty, TSy) \\ &\quad + \lambda^{n(k)-1} d(TSx_0, Tx_0)] \\ &\quad + \lambda^{n(k)} d(TSx_0, Tx_0) + d(TS^{n(k)+1}x_0, Ty), \end{aligned}$$

hence, we get

$$\begin{aligned} d(TSy, Ty) &\leq \frac{a}{1-b}d(Ty, Tx_{n(k)-1}) + \frac{b}{1-b}\lambda^{n(k)-1}d(TSx_0, Tx_0) \\ &\quad + \frac{1}{1-b}\lambda^{n(k)}d(TSx_0, Tx_0) + \frac{1}{1-b}d(Tx_{n(k)+1}, Ty) \\ &\ll \frac{c}{4} + \frac{c}{4} + \frac{c}{4} + \frac{c}{4} = c, \end{aligned}$$

for all $n(k) > N_2$. Thus $d(TSy, Ty) \ll \frac{c}{m}$ for all $m \geq 1$. So, $\frac{c}{m} - d(TSy, Ty) \in P$ for all $m \geq 1$. Since $\frac{c}{m} \rightarrow 0$, as $m \rightarrow \infty$, and P is closed, then $-d(TSy, Ty) \in P$. Therefore, $d(TSy, Ty) = 0$. The injectivity of T imply that $d(Sy, y) = 0$. So S has a fixed point.

- (iii) Let y_1 and y_2 be two fixed points of S . Using the fact that $S \in D_T(a, b)$, the following inequality is satisfied

$$d(Ty_1, Ty_2) = d(TSy_1, TSy_2) \leq ad(Ty_1, Ty_2)$$

which only holds of $Ty_1 = Ty_2$, or since T is injective, $y_1 = y_2$.

- (iv) If T is sequentially convergent, then replacing the sequence $(n(k))_k$ by $(n)_n$ we conclude that $\lim_{n \rightarrow \infty} x_n = y$. Showing in this way that $(x_n)_n$ converges to the fixed point of S .
- (v) Let $x_n \rightarrow p$, where $Sp = p$. Then

$$(3.6) \quad d(TSx_n, TSp) \leq ad(Tx_n, Tp) + bd(Tx_n, TSx_n).$$

Applying the triangle inequality to the left side of inequality above, we obtain

$$0 \leq (1-b)d(TSx_n, Tx_n) \leq (1+a)d(Tx_n, Tp).$$

Since T is continuous, then for $0 \ll c$ given we can choose $N_1 \in \mathbb{N}$ such that

$$d(Tx_n, Tp) \ll \frac{c}{1-b},$$

thus, $d(TSx_n, Tx_n) \ll c$. Therefore, $d(TSx_n, Tx_n) \ll \frac{c}{n}$ for all $n \geq 1$. Then we can conclude that $\frac{c}{n} - d(TSx_n, Tx_n) \in P$ for all $n \geq 1$. Since $\frac{c}{n} \rightarrow 0$, as $n \rightarrow \infty$, and P is closed, we have that $-d(TSx_n, Tx_n) \in P$ which finally imply $\lim_{n \rightarrow \infty} d(TSx_n, Tx_n) = 0$.

- (vi) By the triangle inequality we have

$$d(Tp, TSp) \leq d(Tp, Tx_n) + d(Tx_n, TSx_n) + d(TSx_n, TSp)$$

in view of that $S \in D_T(a, b)$, we get

$$\begin{aligned} d(Tp, TSp) &\leq d(Tp, Tx_n) + d(Tx_n, TSx_n) + ad(Tx_n, Tp) \\ &\quad + b[d(Tx_n, TSx_n) + d(Tp, TSp)], \end{aligned}$$

therefore,

$$0 \leq (1-b)d(Tp, TSp) \leq (1+a)d(Tp, Tx_n) + (1-b)d(Tx_n, TSx_n).$$

From (3.4), reasoning as in part (v), and because T is injective, we have that $Sp = p$.

- (vii) This is an immediate consequence of part (v), equality (3.6), the continuity and injectivity of T .

□

4. FIXED POINT THEOREMS THAT ARE CONSEQUENCE OF THEOREM 3.1

This section is devoted to exhibit how Theorem 3.1 extends the fixed point theorems given in [3, 13, 19, 21, 23, 26, 28, 34]. We would like to put out that the mentioned Theorem 3.1 also generalize the asymptotic results of the cited references.

Fixed point theorems on complete metric spaces. If we take $E = \mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$ and $Tx = x$ in Theorem 3.1, then we obtain (for the corresponding value of a and b) the following:

Corollary 4.1 (E.g. [13], Theorem 2.1). *Let (M, d) be a complete metric space and $S : M \longrightarrow M$ is a contraction mapping. Then S has a unique fixed point.*

Corollary 4.2 (E.g. [19], Theorem 1). *Let (M, d) be a complete metric space and $S : M \longrightarrow M$ a mapping satisfying,*

$$d(Sx, Sy) \leq b[d(x, Sx) + d(y, Sy)],$$

for all $x, y \in M$ and $b \in [0, \frac{1}{2})$. Then S has a unique fixed point.

Corollary 4.3 ([23], Theorem 2). *Let (M, d) be a complete metric space, and $S : M \longrightarrow M$ an arbitrary mapping. If S satisfies the following condition, for $a, b \geq 0$ with $a + 2b < 1$ and any $x, y \in M$,*

$$d(Sx, Sy) \leq ad(x, y) + b[d(x, Sx) + d(y, Sy)].$$

Then S has a unique fixed point.

Now, if we only take $E = \mathbb{R}_+$ in the Theorem 3.1 we obtain:

Corollary 4.4 ([3], Theorem 2.6). *Let (M, d) be a complete metric space and $T : M \longrightarrow M$ be an injective, continuous and subsequentially convergent mapping. Then every continuous function $S : M \longrightarrow M$ satisfying*

$$d(TSx, TSy) \leq ad(Tx, Ty),$$

for all $x, y \in M$ and $0 \leq a < 1$, has a unique fixed point. Moreover, if T is sequentially convergent, then for each $x_0 \in M$, the sequence $(S^n x_0)$ converge to the fixed point of S .

Corollary 4.5 ([26], Theorem 2.1). *Let (M, d) be a complete metric space and $T, S : M \longrightarrow M$ be mappings such that T is continuous, injective and subsequentially convergent. If $b \in [0, \frac{1}{2})$ and*

$$d(TSx, TSy) \leq b[d(Tx, TSx) + d(Ty, TSy)],$$

for all $x, y \in M$, then S has a unique fixed point. Also if T is sequentially convergent, then for each $x_0 \in M$, the sequence $(S^n x_0)$ converge to the fixed point of S .

Next result is the version of $D_T(a, b)$ on metric spaces.

Corollary 4.6. *Let (M, d) be a complete metric space and $T, S : M \longrightarrow M$ be mappings such that T is continuous, injective and subsequentially convergent. If $a, b \geq 0$, with $a + 2b < 1$ and*

$$d(TSx, TSy) \leq ad(Tx, Ty) + b[d(Tx, TSx) + d(Ty, TSy)],$$

for all $x, y \in M$, then S has a unique fixed point. Also if T is sequentially convergent, then for each $x_0 \in M$, the sequence $(S^n x_0)$ converge to the fixed point of S .

Fixed point theorems on complete cone metric spaces. As in previous part, for the corresponding different values of a and b , if we assume $Tx = x$ in Theorem 3.1 we get:

Corollary 4.7 ([34], Theorem 2.3). *Let (M, d) be a complete cone metric space and the mapping $S : M \longrightarrow M$ is a contraction function. Then S has a unique fixed point in M and for any $x_0 \in M$, we have that $(S^n x_0)$ converges to the fixed point.*

Corollary 4.8 ([34], Theorem 2.6). *Let (M, d) be a complete cone metric space and the mapping $S : M \longrightarrow M$ satisfying,*

$$d(Sx, Sy) \leq b[d(x, Sx) + d(y, Sy)],$$

for all $x, y \in M$ and $b \in [0, \frac{1}{2})$. Then S has a unique fixed point and for any $x_0 \in M$, we have that $(S^n x_0)$ converges to the fixed point.

Corollary 4.9 (C.f. [21], Theorem 2.7). *Let (M, d) be a complete cone metric space and $S : M \longrightarrow M$ an arbitrary mapping. If S satisfies the following condition, for $a, b \geq 0$ with $a + 2b < 1$ and any $x, y \in M$,*

$$d(Sx, Sy) \leq ad(x, y) + b[d(x, Sx) + d(y, Sy)].$$

Then S has a unique fixed point.

The final case is to consider $T \neq \text{id}$, where by id we denote the identity function $\text{id}x = x$ for all $x \in M$.

Corollary 4.10 ([28], Theorem 3.1). *Let (M, d) be a complete cone metric space, $T : M \longrightarrow M$ be an one to one and continuous function, moreover let $S : M \longrightarrow M$ a continuous function satisfying*

$$d(TSx, TSy) \leq ad(Tx, Ty),$$

for all $x, y \in M$ and $0 \leq a < 1$. Then

(i) *For every $x_0 \in M$,*

$$\lim_{n \rightarrow \infty} d(TS^n x_0, TS^{n+1} x_0) = 0;$$

(ii) *There is $y_0 \in M$ such that*

$$\lim_{n \rightarrow \infty} TS^n x_0 = y_0;$$

(iii) *If T is subsequentially convergent, then $(S^n x_0)$ has a convergent subsequence;*

(iv) *There is a unique $z_0 \in M$ such that*

$$Sz_0 = z_0;$$

(v) *If T is sequentially convergent, then for each $x_0 \in M$ the iterate sequence $(S^n x_0)$ converges to z_0 .*

The following result generalize the T -Kannan class given in [26].

Corollary 4.11. *Let (M, d) be a complete cone metric space and $T, S : M \longrightarrow M$ be mappings such that T is continuous, injective and subsequentially convergent. If $b \in [0, \frac{1}{2})$, and*

$$d(TSx, TSy) \leq b[d(Tx, TSx) + d(Ty, TSy)],$$

for all $x, y \in M$, then S has a unique fixed point. Also if T is sequentially convergent, then for each $x_0 \in M$, the sequence $(S^n x_0)$ converge to the fixed point of S .

Finally, we would like summarize these previous results in the following table: Notice that each box in the table below indicate the reference of the theorem generalized by Theorem 3.1 for each case of the function T in the left upper corner.

$T = \text{id}$ $T \neq \text{id}$	$a, b \geq 0$	$a = 0$ $b \in [0, \frac{1}{2})$	$b = 0$ $a < 1$
(M, d) be a complete cone metric space	C.f., [21, Thm 2.7] $D_T(a, b)$	[34, Thm 2.6] $D_T(0, b)$	[34, Thm 2.3] [28, Thm 3.1]
$E = \mathbb{R}$	[23, Thm 2] $D_T(a, b)$	[19, Thm 1] [26, Thm 2.1]	[13, Thm 2.1] [3, Thm 2.6]

TABLE 1. Results generalized by Theorem 3.1.

REFERENCES

- [1] M. Abbas and G. Jungck, *Common fixed point results for noncommuting mappings without continuity in cone metric spaces*, J. Math. Anal. Appl. 341, (1), (2008), 416–420.
- [2] M. Abbas and B. E. Rhoades, *Fixed and periodic results in cone metric spaces*, Appl. Math. Lett., 22, (2009), 511–515.
- [3] A. Beiranvand, S. Moradi, M. Omid and H. Pazandeh, *Two fixed point theorem for special mapping*, arXiv: 0903.1504v1 [math.FA].
- [4] V. Berinde, *Generalized contractions in uniform spaces*, Bul. Stiint. Univ. Baia Mare, (9), (1993), 45–52.
- [5] V. Berinde, *A fixed point theorem of Maia type in K-metric spaces*, Seminar on Fixed Point Theory, 3, (1991), 7–14.
- [6] A. Branciari, *A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces*, Publi. Math. Debrecen, (57), 1–2, (2000), 31–37.
- [7] S.K. Chatterjea, *Fixed point theorem*, C.R. Acad Bulgar Sci. 25 (1972). 727–730.
- [8] L.B. Ćirić, *A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc, (45), (1974), 267–273.
- [9] P. Collaço and J. Carvalho e Silva, *A complete comparison of 25 contraction conditions*, Nonlinear Analysis, Theory, Methods and Applications, Vol 30, 471–176. Proc. 2nd World Congress of Nonlinear Analysts, 1997.
- [10] G. Das and J.P. Debata, *Fixed point of quasi-nonexpansive mappings*, Indian J. Pure Appl. Math., 17, (1986), 1263–1269.
- [11] W.R. Derryck and L. Nova, *Interior properties and fixed points of certain discontinuous operators*, Progress in Functional Analysis, Proc. Int. Meet. Occas. 60th Birthd. M. Valdivia, Spain, North-Holland, Math. Stud., 170, (1992), 239–245.
- [12] W.R. Derryck and L. Nova, *Fixed point theorems for discontinuous operators*, Glasnik Matematički, (1989), 339–347.
- [13] K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge University Press, NY, 1990.
- [14] K. Goebel, W.A. Kirk and T.N. Chimi, *A fixed point theorem in uniformly convex spaces*. Boll. Union. Math. Ital., 7, (1973), 67–75.
- [15] G.E. Hardy and T.D. Rogers, *A generalization of a fixed point theorem of Reich*, Canad. Math. Bull., 16, (1973), 201–206.
- [16] D. Ilic and V. Rakocevic, *Common fixed point for maps on cone metric space*, J. Math. Anal. Appl. 341, (2), (2008), 876–882.
- [17] V. Istrăţescu, *Fixed Point Theory, an Introduction*, Mathematics and Its Applications /7, Boston U.S.A, D. Reidel Publishing Company, 2002.

- [18] G. Jungck, S. Radenovic, S. Radojevic and V. Rakocevic, *Common fixed point theorems for weakly compatible pairs on cone metric spaces*, Fixed Point Theory and Applications, vol 2009, (2009), art. ID: 643840, 13p.
- [19] R. Kannan, *Some results on fixed points II*, Amer. Math. Monthly, 76, (1969), 405–408.
- [20] R. Kannan, *Some results on fixed points III*, Fund. Math., 70, (1971), 169–177.
- [21] M.S. Khan and M. Samanipour, *Fixed point theorems for some discontinuous operators in cone metric space*, Mathematica Moravica, Vol **12-2**, (2008), 29–34.
- [22] H. Long - Guang and Z. Xian, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl. 332 (2007), 1468–1476.
- [23] L. Nova, *Fixed point theorems for some discontinuous operators*, Pacific Journal of Maths, (1986), 189–196.
- [24] H. Mohebi, *Topical functions and their properties in a class of ordered Banach spaces*, in: Continuous Optimization Current Trends and Modern Applications, Part II, Springer, 2005, pp. 343–361.
- [25] H. Mohebi, H. Sadeghi and A.M. Rubinov, *Best approximation in a class of normed spaces with star-shaped cone*, Numer. Funct. Anal. Optim. 27, (**34**), (2006), 411–436.
- [26] S. Moradi, *Kannan fixed-point theorem on complete metric spaces and on generalized metric spaces depended an another function*, arXiv: 0903.1577v1 [math.FA].
- [27] R. Morales and E. Rojas, *Some results for one class of discontinuous operators with fixed points*, Acta. Math. Univ. Comenianae, Vol. LXXVI, 2, (2007), 149–160.
- [28] J.R. Morales and E. Rojas, *Cone metric spaces and fixed point theorems of T-contractive mappings*, preprint, (2009), 14 pp.
- [29] P. Raja and S. M. Vaezpour, *Some extensions of Banach’s contraction principle in complete cone metric spaces*, fixed point theory and applications, (2008), 11 p.
- [30] D. Rani and R. Chugh, *Some fixed point theorems on contractive type mappings*, Pure Appl. Math. Sci. (**39**), 1–2, (1994), 153–158.
- [31] S. Reich, *Kannan’s fixed point theorems*, Boll. Union. Math. Ital., 4, (1971), 121–128.
- [32] S. Reich, *Remarks on fixed points*, Rend. Accad. Naz. di lincei., 52, (1972), 689–697.
- [33] S. Reich, *Some problems and results in fixed point theory*, Contemp. Math., (**21**), (1983), 179–187.
- [34] Sh. Rezapour and R. Hamlbarani, *Some notes on the paper “Cone metric spaces and fixed point theorems of contractive mappings”*, J. Math. Anal. Appl. 345, (**2**), (2008), 719–724.
- [35] B. E. Roades, *A comparison of various definitions of contractive mappings*, Trans. Amer. Math. Soc., 226, (1977). 257–290.
- [36] B. E. Roades, *Contractive definitions revisited*, Contemp. Math., (**21**), (1983), 179–187.
- [37] B. E. Roades, *Contractive definitions*, Nonlinear Analysis (ed. T. M. Rassias) World Scientific Publishing Company, New Jersey, (1988), 513–526.
- [38] P.P. Zabrejko, *K-metric and K-normed linear spaces: survey*, Collect. Math. **48**, 4-6, (1997), 825–859.